# Analytic Solution for CVA of a Collateralised Call Option

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**Abstract**

We adapt the methodology proposed by Turfus (2016a) for the analytic pricing of CoCo bonds under a Black-Karasinski credit default intensity model to calculate the CVA on a vanilla European equity option with a deterministic collateral schedule, incorporating correlation risk between the equity and the option writer and a possible jump in the equity price upon default of the option writer. We also extend our analysis to cover the case when the collateral is updated to reflect the option value on a predetermined schedule. We see that our results are equally applicable to pricing the CVA of a portfolio of options.

## 1 Introduction

We look to obtain analytical formulae for the counterparty value adjustment (CVA) for vanilla European put and call options on an equity underlying. We take CVA to be defined in the standard manner as described, for example, in Hull and White (2012) and Brigo et al. (2013).

This problem is considered in §8.4 of Brigo et al. (2013). The credit model they used was a structural one with assumed correlation between the firm values of the equity issuer and of the option counterparty. Zero collateral level was assumed. Results were obtained via Monte Carlo simulation for various levels of the correlation, which was seen to have a significant impact on the CVA: a correlation of $\pm 25\%$ can have the effect of doubling or halving the CVA.

Brigo et al. (2013) also consider in §5.3 the CVA on interest rate swap portfolios and swaptions. A reduced credit model was assumed for counterparty default risk, based on the Cox, Ingersoll and Ross (1985) short rate model. The interest rate model they used was a two-factor Hull and White (1990) short rate model. A correlation was assumed between the diffusions of the respective models. Again they calculated the CVA via Monte Carlo simulation. They also report in §4.4 analytical results for the swap portfolio case, based on swaption modelling and a "drift freezing" approximation, but only under the assumption of zero rates-credit correlation.

We work with a reduced credit model for the option counterparty, supposing that the default intensity $\lambda_t$ is governed by a Black and Karasinski (1991) short rate model. The equity price is taken to be given by a jump-diffusion process, with a downward jump of a fixed relative amount occurring at the time of default. The diffusive processes are potentially correlated. We assume from the outset that interest rates are deterministic, this being a common assumption in equity option pricing.

The modelling of the equity and credit default intensity underlyings is presented in §2.1. We consider first the result for a call option, presenting in §2.2 the payoff expression and that for the loss on default. The equation governing the dynamics of the CVA value is derived in §2.3 supposing in the first instance that the collateral level follows a predetermined schedule over the life of the option. The solution for the CVA is then obtained using a perturbation expansion approach, as described in §3. It is shown in §4 how the approach can be extended straightforwardly to put options and, with a bit more effort, to obtaining similar expressions in the case when the collateral value is updated at regular intervals to reflect the current value of the option. It is also elucidated how a portfolio of options under a netting agreement can be handled analogously. Finally some directions for future work are presented in the closing section §5

# 2 Mathematical Model

## 2.1 Basic Framework

We take the credit default intensity to be governed by a Black-Karasinski short rate model. We shall find it convenient to work not with $\lambda_t$ directly, but with an auxiliary process $y_t$ satisfying the following canonical Ornstein-Uhlenbeck process:

$$dy_t = -\alpha y_t dt + \sigma_y(t)\, dW_t^1 \tag{1}$$

with $y_0 = 0$, where $\alpha$ is a positive constant specifying the mean reversion rate,[1] $\sigma_y(t)$ is a bounded positive $L^2$ function and $dW_t^1$ is a Brownian motion for $t \in D_m := [0, \min\{\tau, T_m\}]$ where $\tau$ is the time of default of the option writer and $T_m$ is the maturity date of the option under consideration. This equation is well known to have a unique strong solution subject to these assumptions. This auxiliary variable is related to the credit default intensity $\lambda_t$ by

$$\lambda_t = (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}(y_t), \tag{2}$$

where $\overline{\lambda}(t)$ is the credit spread defined by

$$E\left[e^{-\int_0^t \lambda_s\, ds}\right] = e^{-\int_0^t \overline{\lambda}(s)\, ds} \tag{3}$$

and $\mathcal{E}(X_t) := \exp\left(X_t - \frac{1}{2}[X]_t\right)$ is a stochastic exponential with $[X]_t$ the quadratic variation of a process $X_t$. The required form of the configurable function $\lambda^*(t)$ is determined by calibration of the model to satisfy the no-arbitrage condition set out below. We see immediately from differentiating Eq. (3) and setting $t = 0$ that $\lambda_0 = \overline{\lambda}(0)$ whence we deduce $\lambda^*(0) = 0$.

For the equity process we propose:

$$\frac{dS_t}{S_t} = \begin{cases} (\overline{r}(t) - q(t) - k\lambda_t)\, dt + \sigma_1(t)\, dW_t^2 + k dn_t & \text{if } t \leq \tau, \\ (\overline{r}(t) - q(t))\, dt + \sigma_2(t)\, dW_t^2 & \text{if } t > \tau, \end{cases} \tag{4}$$

where $S_t$ is the equity price, $q(t)$ its expected (continuous) dividend rate, $\overline{r}(t)$ the instantaneous forward rate of interest, $dW_t^2$ a Brownian motion satisfying

$$\mathrm{corr}(W_t^1, W_t^2) = \rho_{\lambda S} \tag{5}$$

and $n_t$ a Cox process with intensity $\lambda_t$ giving rise to an equity price jump of size $k$ with $-1 < k \leq 0$, contingent on default. We denote by $\mathcal{F}^W$ the filtration generated by $W_t^1$ and $W_t^2$ and by $\mathcal{F}^J$ that generated by $n_t$. We further define a combined filtration $\mathcal{F} = \mathcal{F}^W \vee \mathcal{F}^J$ and formally assume the hypothesis $\mathcal{H}$ of Ehlers and Schönbucher (2006) under which $\mathcal{F}$- and $\mathcal{F}^W$-martingales can be considered equivalent.

We also define a pre-default equity term variance

$$I_1(t_1, t_2) := \int_{t_1}^{t_2} \sigma_1^2(u) du. \tag{6}$$

If a default occurs at time $v$ with $t_1 < v < t_2$, the equity term variance (ignoring the jump impact) will reflect the different equity dynamics in Eq. (8) pre- and post-default and be given by

$$I_2(t_1, v, t_2) := \int_{t_1}^{v} \sigma_1^2(u) du + \int_{v}^{t_2} \sigma_2^2(u) du. \tag{7}$$

When we have $t_1 = v$ in Eq. (7), we shall write, for convenience, $I_2(v, t_2) := I_2(v, v, t_2)$.

[1] It is a straightforward matter to loosen this assumption and specify instead that $\alpha(t)$ be a bounded positive $L^1$ function, whereupon the analysis below goes through effectively replacing $\alpha(v-u)$ throughout by $\int_u^v \alpha(s) ds$.

As with the credit default intensity, it is convenient to express the equity price in terms of a new auxiliary process $x_t$, conditional on default at time $\tau = v$ through

$$S_t = \begin{cases} F_1(t) e^{x_t - \frac{1}{2} I_1(0,t)} & \text{if } t < v, \\ (1+k) F_2(v,t) e^{x_t - \frac{1}{2} I_2(0,v,t)} & \text{if } t \geq v, \end{cases} \tag{8}$$

where

$$F_1(t) := S_0 e^{\int_0^t (\overline{r}(s) - q(s) - k\overline{\lambda}(s)) ds}, \tag{9}$$

$$F_2(v,t) := F_1(v) e^{\int_v^t (\overline{r}(s) - q(s)) ds}, \tag{10}$$

noting that by construction $x_0 = 0$. We define also for convenience

$$M_1(x,t) := e^{x - \frac{1}{2} I_1(0,t)} F_1(t), \tag{11}$$

$$M_2(x,v,t) := e^{x - \frac{1}{2} I_2(0,v,t)} F_2(v,t). \tag{12}$$

and risk-free and risky discount functions

$$D(t_1,t_2) := e^{-\int_{t_1}^{t_2} \overline{r}(s) ds}, \tag{13}$$

$$B(t_1,t_2) := e^{-\int_{t_1}^{t_2} (\overline{r}(s) + \overline{\lambda}(s)) ds}. \tag{14}$$

To allow the model to be calibrated we suppose that a term structure of risky bond prices $B(0,t_i)$ is known for $t_1 < t_2 < \ldots < t_n$. The required no-arbitrage condition is then that

$$D(0,t_i) E\left[e^{-\int_0^{t_i} \lambda_s \, ds}\right] = B(0,t_i), \quad i = 1,2,\ldots,n. \tag{15}$$

Eqs. (3) and (15) taken together with an interpolation scheme for times between the nodes $t_i$ allow in principle both $\overline{\lambda}(t)$ and $\lambda^*(t)$ to be determined for $t \leq t_n$.

## 2.2 Payoff Definitions

For the CVA calculation, we have two payoffs to consider. In the first instance there is the option payoff at at its maturity date, $T_m$ say, conditional on no default. Secondly, there is a loss on default at time $\tau$ if $\tau < T_m$, driven by the value of the option payoff. The latter gives rise to the CVA. We shall consider these payoffs in turn.

In practice we shall need the value of the option not at the initial time 0 but at the default time $\tau = v$, say. To that end the calculation of the option value will be based on the second line of Eq. (4). Let us suppose that $x_v = \xi$ and $y_v = \eta$. The equity value pre-jump at $v^-$ will be determined by the *first* line of Eq. (8), whence that immediately *after* the jump has occurred will be given by $(1+k)M_1(\xi,v) \equiv (1+k)M_2(\xi,v,v)$. On this basis the call option price will be given by the standard Black formula, in the present notation:

$$f_m(\xi,v) = D(v,T_m)\left((1+k)M_2(\xi,v,T_m)N(d_1(\xi,v,T_m)) - KN(d_2(\xi,v,T_m))\right), \tag{16}$$

where

$$d_2(x,t,u) := \frac{\ln((1+k)M_2(x,t,u)) - \ln K}{\sqrt{I_2(t,u)}}, \tag{17}$$

$$d_1(x,t,u) := d_2(x,t,u) + \sqrt{I_2(t,u)} \tag{18}$$

and $N(\cdot)$ as usual represents a standard cumulative normal distribution.

In considering the CVA payoff at time $v$, suppose that a predetermined schedule governs the collateral level and denote this by $C(t)$. The CVA loss calculated at $v$ will be given by $1 - R$ times the difference

between the option value and this collateral level if positive, zero otherwise, where R is the assumed recovery level of the counterparty. Thus the payoff function for default at time $\tau = v$ is given by

$$P_m(\xi, v) = (1 - R) \max \{ f_m(\xi, v) - C(v), 0 \} . \tag{19}$$

The counterparty value adjustment at time $t \geq 0$ is then defined in our notation to be

$$\begin{aligned} \mathrm{CVA}(t) &= (1 - R) \int_t^{T_m} D(t, v) E \left[ \mathbb{1}_{\tau \geq v} P_m(x_v, v) \middle| \mathcal{F}_t \right] d\mathbb{P}(v) \\ &= (1 - R) \int_t^{T_m} D(t, v) E \left[ \lambda_v e^{-\int_t^v \lambda_s \, ds} P_m(x_v, v) \middle| \mathcal{F}_t^W \right] dv, \end{aligned} \tag{20}$$

where $\mathbb{P}(\cdot)$ is the probability distribution of the time of default events driven by the Cox process $n_t$ defined above and in deriving the last line we have relied upon the result of Ehlers and Schönbucher (2006).

### 2.3 Main Equation

We next suppose that the process $\mathrm{CVA}(t)$ can be expressed in terms of the new (stochastic) variables $x_t$ and $y_t$ as $h(x_t, y_t, t)$. We are interested in calculating $\mathrm{CVA}(0) = h(0, 0, 0)$. Applying the well-known Feynman-Kac method to Eq. (20) subject to Eqs. (1), (4) and (5), we infer that the function $h(x, y, t)$ representing the CVA is governed by the following backward diffusion equation:

$$\mathcal{L}[h(x, y, t)] = - \left( \overline{\lambda}(t) + \Delta\lambda(y, t) \right) P_m(x, t) + \Delta\lambda(y, t) \left( h + k \frac{\partial h}{\partial x} \right) \tag{21}$$

for $t \in D_m$ with final condition that $h(x, y, T_m) = 0$, where $\mathcal{L}[\cdot]$ is a forced diffusion operator given by

$$\mathcal{L}[\cdot] := \frac{\partial}{\partial t} - \alpha y \frac{\partial}{\partial y} + \frac{1}{2} \left( \sigma_1^2(t) \frac{\partial^2}{\partial x^2} + 2\rho_{\lambda S} \sigma_1(t) \sigma_y(t) \frac{\partial^2}{\partial x \partial y} + \sigma_y^2(t) \frac{\partial^2}{\partial y^2} \right) - (\overline{r}(t) + \overline{\lambda}(t)) \tag{22}$$

and

$$\Delta\lambda(y, t) := (\overline{\lambda}(t) + \lambda^*(t)) \mathcal{E}_y(y_t) - \overline{\lambda}(t), \tag{23}$$

where $\mathcal{E}_y(y_t)$ for $y \in \mathbb{R}$ is a shorthand for $\mathcal{E}(y_t)|_{y_t = y}$. Our task is then to solve Eq. (21) to obtain $\mathrm{CVA}(0) = h(0, 0, 0)$.

## 3 Perturbation Analysis

In the absence of exact closed form solutions to Eq. (21), we seek an approximate solution for $\mathrm{CVA}(t)$ under the low default intensity assumption proposed by Turfus (2016b), defining an asymptotically small parameter $\epsilon$ by

$$\epsilon := \frac{1}{\alpha T_m} \int_0^{T_m} \overline{\lambda}(t) dt.$$

We then define a scaled forward intensity $\tilde{\lambda}(t)$ by

$$\tilde{\lambda}(t) := \epsilon^{-1} \overline{\lambda}(t) \tag{24}$$

and further define

$$\Delta\tilde{\lambda}(y, t) = \epsilon^{-1} \Delta\lambda(y, t) \tag{25}$$

taking these to be $O(1)$ as $\epsilon \to 0$. Turfus (2016b) shows that we can write

$$\lambda^*(t) = \epsilon^2 \lambda_2^*(t) + O(\epsilon^3).$$

The form of $\lambda_2^*(t)$ is not required for the present purposes so need not concern us further here.

To solve Eq. (21) in it asymptotic representation we pose a perturbation expansion

$$h(x,y,t) = \epsilon h_1(x,y,t) + \epsilon^2 h_2(x,y,t) + O(\epsilon^3). \tag{26}$$

Substituting the above expansion into Eq. (21), solving up to $O(\epsilon)$, setting $t = 0$ and reverting to unscaled notation, we obtain the following first order accurate result:

**Theorem 3.1** *The CVA on a European call option on an equity underlying can be estimated under our modelling assumptions as follows:*

$$\begin{aligned} CVA(0) = {} & (1-R)\int_0^{T_m} B(0,v)D(v,T_m)\overline{\lambda}(v)(1+k)F_2(v,T_m)e^{I_\rho(0,v)}\psi_1^+(v,T_m)\,dv \\ & -(1-R)\int_0^{T_m} B(0,v)\overline{\lambda}(v)\left(D(v,T_m)K\psi_2^+(v,T_m) + C(v)N(a_2(I_\rho(0,v),0,v))\right)dv \\ & + O(\epsilon^2) \end{aligned} \tag{27}$$

*where*

$$\begin{aligned} I_\rho(t_1,t_2) &:= \rho_{\lambda S}\int_{t_1}^{t_2} e^{-a(t_2-u)}\sigma_y(u)\sigma_1(u)du, & (28)\\ \psi_i^{\pm}(v,w) &:= N_2(\pm a_i(I_\rho(0,v),0,v), \pm b_i(I_\rho(0,v),0,v,w), R(0,v,w)), \quad i=1,2, & (29)\\ a_2(x,t,v) &:= \frac{x-\xi^*(v)}{\sqrt{I_1(t,v)}}, & (30)\\ a_1(x,t,v) &:= a_2(x,t,v) + \sqrt{I_1(t,v)}, & (31)\\ b_2(x,t,v,w) &:= \frac{\ln((1+k)M_2(x,v,w)) - \ln K}{\sqrt{I_2(t,v,w)}}, & (32)\\ b_1(x,t,v,w) &:= b_2(x,t,v,w) + \sqrt{I_2(t,v,w)}, & (33)\\ R(t,v,w) &:= \sqrt{\frac{I_1(t,v)}{I_2(t,v,w)}}, \quad v<w, & (34)\\ \xi^*(v) &:= \sup\{\xi \,|\, P_m(\xi,v) = 0\}, & (35) \end{aligned}$$

*and $N_2(x_1,x_2;\rho)$ is a standard bivariate cumulative Gaussian distribution with correlation $\rho$.*

**Proof.** For the proof of Theorem 3.1, see Appendix A. Note that in terms of our previous notation $b_i(x,v,v,w) \equiv d_i(x,v,w)$, $i = 1,2$. □

The corresponding result for the CVA on a put option is obtained similarly.

**Theorem 3.2** *The CVA on a European put option on an equity underlying can be estimated under our modelling assumptions as follows:*

$$\begin{aligned} CVA(0) = {} & (1-R)\int_0^{T_m} B(0,v)\overline{\lambda}(v)\left(D(v,T_m)K\psi_2^-(v,T_m) - C(v)N(-a_2(I_\rho(0,v),0,v))\right)dv \\ & -(1-R)\int_0^{T_m} B(0,v)\overline{\lambda}(v)(1+k)D(v,T_m)F_2(v,T_m)e^{I_\rho(0,v)}\psi_1^-(v,T_m)\,dv + O(\epsilon^2). \end{aligned} \tag{36}$$

*The notation is as defined for Theorem 3.1, except in that the option price must now be reinterpreted as*

$$f_m(\xi,v) = D(v,T_m)\left(KN(-d_2(\xi,v,T_m)) - (1+k)M_2(\xi,v,T_m)N(-d_1(\xi,v,T_m))\right) \tag{37}$$

*in the definition of $P_m(\xi,v)$, rather than as given for the call option in Eq.* (16).[2]

[2] We must also reinterpret $\xi^*(v)$ as inf rather than sup contingent on this reinterpretation of notation.

**Proof.** The proof follows closely that of Theorem 3.1 in Appendix A, making use of the formula for a compound call option on a put option provided in Geske (1979). $\square$

In interpreting the significance of these results, we note that the main impact of correlation will be through the appearance of the exponential involving $I_\rho(\cdot)$ multiplying the equity forward. If the equity and credit intensity are assumed *negatively* correlated so that $I_\rho(0, v) < 0$, the equity price at default will be *less* than otherwise assumed. The value of the call option at default will in this way be reduced, whence the risk is inferred to be "right-way" (and correspondingly "wrong-way" for the put). Concerning the impact of the (downwards) jump at default, the main consequence of this will be further to decrease the value of the equity at default so will impact in the same direction as the assumed negative correlation. For short-term options, the jump effect will in likelihood be the more important: the impact on world stock markets from the failure of Lehman Brothers was around 3–4% downwards. But as the impact of correlated diffusion on the option price will accumulate over time with, by our assumption, no compensating adjustment made to the collateral level, for longer maturity options, this effect will ultimately exceed the jump impact and become the more important.

# 4 Extensions of Main Result

## 4.1 Dynamic Collateral

It is of course often the case in practice that collateral requirements are not fixed at the onset of the trade but change to reflect the level of exposure. In this section we look to see how the results of Theorems 3.1 and 3.2 can be extended to take account of a policy whereby at scheduled dates $t_j$, $j = 0, 1, \ldots, n-1$ the collateral is updated to reflect the current exposure level associated with the option, viz. its PV. Clearly in this situation, each exposure period $[t_j, t_{j+1})$ can be considered separately, with the compound option taken to be forward-starting at $t_j$ with strike given by the value at time $t_j$ of the option, conditional on survival of the option writer.[3] To that end we need to take the market view of the option value at time $t \geq t_j$ conditional on survival to time $t_j$. We argue that this can be achieved by considering the equity dynamics to be given by the first line of Eq. (4) until $t_j$ and by the second line thereafter. Thus we propose

$$S_t = F_2(t_j, t) e^{x_t - \frac{1}{2} I_2(0, t_j, t)}, \quad t \geq t_j \geq 0$$

By straightforward calculation, we infer the market value of the option as of time $t_j$ conditional on $x_{t_j} = x$ and $\tau > t_j$ is

$$C_j(x) = D(t_j, T_m)(M_2(x, t_j, T_m) N(d_1^*(x, t_j, T_m)) - K N(d_2^*(x, t_j, T_m))), \tag{38}$$

where

$$d_2^*(x, t, u) := \frac{\ln M_2(x, t, u) - \ln K}{\sqrt{I_2(t, u)}}, \tag{39}$$

$$d_1^*(x, t, u) := d_2^*(x, t, u) + \sqrt{I_2(t, u)} \tag{40}$$

The payoff function we must now consider conditional on $x_{t_j} = x_j$ for $j = 0, 1, \ldots, n-1$ becomes

$$P_m^*(\xi, v) := (1 - R) \max\{f_m(\xi, v) - C_j(x_j), 0\} (H(v - t_j) - H(v - t_{j+1})), \tag{41}$$

with $H(\cdot)$ the Heaviside step function. In this way we obtain:

[3] We have ignored the possibility that $t_0 \neq 0$, viz. the situation where the collateral for the first period is already known rather than being determined dynamically. However this is the case already considered in Theorem 3.1 so can be dealt with in the manner described there, with the caveat that $t_1$ rather than $T_m$ must be used as the upper limit in the integrals.

**Theorem 4.1** *The CVA on a European call option on an equity underlying associated with default in a forward time period $[t_j, t_{j+1})$ where the collateral is reset at $t_j$ can be estimated under our modelling assumptions and the approximation Eq. (55) as follows:*

$$\begin{aligned}
CVA^{(j)}(0) \approx (1-R)&\int_{t_j}^{t_{j+1}} B(0,v)D(v,T_m)\overline{\lambda}(v)(1+k)F_2(v,T_m)e^{I_\rho(0,v)}\psi_1^{(j)+}(I_\rho(0,v),v,T_m)\,dv \\
&- (1-R)K\int_{t_j}^{t_{j+1}} B(0,v)D(v,T_m)\overline{\lambda}(v)\psi_2^{(j)+}(I_\rho(0,v),v,T_m)\,dv \\
&- (1-R)\int_{t_j}^{t_{j+1}} \overline{\lambda}(v)B(0,v)D(v,T_m)N\left(a_2^{(j)}(I_\rho(t_j,v),t_j,v)\right)dv \\
&\quad \left(e^{I_\rho(0,t_j)}F_2(t_j,T_m)N(b_1^{(j)}(I_\rho(0,t_j),0,T_m)) - KN(b_2^{(j)}(I_\rho(0,t_j),0,T_m))\right),
\end{aligned} \tag{42}$$

*while the corresponding result for the CVA on a European put option is*

$$\begin{aligned}
CVA^{(j)}(0) \approx (1-R)K&\int_{t_j}^{t_{j+1}} B(0,v)D(v,T_m)\overline{\lambda}(v)\psi_2^{(j)-}(I_\rho(0,v),v,T_m)\,dv \\
&- (1-R)\int_{t_j}^{t_{j+1}} B(0,v)D(v,T_m)\overline{\lambda}(v)e^{I_\rho(0,v)}(1+k)F_2(v,T_m)\psi_1^{(j)-}(I_\rho(0,v),v,T_m)\,dv \\
&- (1-R)\int_{t_j}^{t_{j+1}} \overline{\lambda}(v)B(0,v)D(v,T_m)N\left(-a_2^{(j)}(I_\rho(t_j,v),t_j,v)\right)dv \\
&\quad \left(KN(-b_2^{(j)}(I_\rho(0,t_j),0,T_m)) - e^{I_\rho(0,t_j)}F_2(t_j,T_m)N(-b_1^{(j)}(I_\rho(0,t_j),0,T_m))\right),
\end{aligned} \tag{43}$$

*where*

$$\psi_i^{(j)\pm}(v,w) := N_2(\pm a_i^{(j)}(I_\rho(0,v),0,v), \pm b_i^{(j)}(I_\rho(0,v),0,v,w), R(0,v,w)), \quad i=1,2, \tag{44}$$

$$a_2^{(j)}(x,t,v) := \frac{x-\xi_j^*(v)}{\sqrt{I_1(t,v)}}, \tag{45}$$

$$a_1^{(j)}(x,t,v) := a_2(x,t,v) + \sqrt{I_1(t,v)}, \tag{46}$$

$$b_2^{(j)}(x,t,w) := \frac{\ln M_2(x,t_j,w) - \ln K}{\sqrt{I_2(t,t_j,w)}}, \tag{47}$$

$$b_1^{(j)}(x,t,w) := b_2(x,t,w) + \sqrt{I_2(t,t_j,w)} \tag{48}$$

*and the definition of $\xi_j^*(v)$ is given in Appendix B below.*

**Proof.** For the proof of Theorem 4.1, see Appendix B. □

The total CVA for the option is then simply the sum over $j$ of all contributions of the form of Eq. (42) or (43). Cancellation between terms means that the impact of correlation is considerably reduced in comparison with the deterministic collateral case considered in Theorems 3.1 and 3.2 above. Basically the impact of correlation on CVA only has the chance to build up between collateral readjustment dates, when it is reset to zero. However, the impact of the jump is expected to stay as strong as previously, so will tend always be the greater of the two effects.

## 4.2 Portfolio of Options

The extension of the above result to a portfolio of options on a single underlying, potentially combining calls or puts with various strikes and maturities is not difficult. We assume here that the collateral for all options

is updated on a single schedule. We further assume that a collateral adjustment occurs on the expiry dates of all the options. Rather than using the call option payoff of $\max\{f_m(\xi, v) - f_m(x_j, t_j), 0\}$ in Eq. (41), we must use a sum over all the options in the portfolio, namely

$$\max\left\{\sum_i \left(f_m^{(i)}(\xi, v) - C_j^{(i)}(x_j)\right), 0\right\}$$

where the superscript $(i)$ denotes the $i$th option in the portfolio. Reinterpreting $\xi^*(x_j, v) = x_j + \xi^*(v)$ as relating to this aggregated payoff function for a given $v \in [t_j, t_{j+1})$, the analysis goes through identically to obtain the CVA of each call option separately using the same $\xi^*(v)$ in all cases; similarly for a portfolio of puts. If there are both calls and puts, it is expected to be the case that the payoff will be in the money if $\xi$ is high enough or low enough. So there will be two critical values, each giving a contribution to $\mathrm{CVA}^{(j)}(0)$ when its value is exceeded. There is nothing to prevent the inclusion either of both long and short positions in the portfolio, provided it is the case that collateral is posted to the counterparty (as usually would happen) should the total value of the portfolio ever become negative.

# 5 Future Work

It has not yet been possible to assess quantitatively the relative importance of the correlation and jump correction terms proposed above for the various cases considered, nor to demonstrate the accuracy of the assumed first order expansions or the impact of the approximation Eq. (55). This is the subject of ongoing research work by the author.

It should also be possible to extend these results fairly straightforwardly to other European equity options such as digitals. Similarly the equity underlying could be swapped for an FX rate, an inflation rate or a commodity spot price.

# A Proof of Theorem 3.1

We derive Eq. (27), solving Eq. (21) in the standard manner by successive levels of approximation. At first order we must solve

$$\mathcal{L}[h_1(x, y, t)] = -\tilde{\lambda}(t)\mathcal{E}_y(y_t)P_m(x, t)$$

for $t \in D_m$ with final condition that $h_1(x, T_m) = 0$. This can be achieved by means of the following readily obtainable Green's function for the diffusion operator $\mathcal{L}[\cdot]$:

$$G(x, y, t; \xi, \eta, v) = B(t, v)\frac{\partial^2}{\partial\xi\partial\eta}N_2\left(\frac{x - \xi}{\sqrt{I_1(t, v)}}, \frac{ye^{-\alpha(v-t)} - \eta}{\sqrt{I_y(t, v)}}; \rho(t, v)\right), \quad 0 \le t \le v, \tag{49}$$

where $B(t, v)$ is the risky discount factor defined in Eq. (14) above,

$$\rho(t, v) := \frac{I_\rho(t, v)}{\sqrt{I_1(t, v)I_y(t, v)}}, \tag{50}$$

$$I_y(t_1, t_2) := \int_{t_1}^{t_2} e^{-2\alpha(t_2 - u)}\sigma_y^2(u)du \tag{51}$$

and $I_1(t_1,t_2)$ and $I_\rho(t_1,t_2)$ are as defined in Eqs. (6) and (28) above, respectively. Making use of Eq. (49) we obtain

$$
\begin{aligned}
h_1(x,y,t) &= (1-R)\int_t^{T_m} \overline{\lambda}(v) \int_{\xi^*(v)}^{\infty} \Big(f_m(\xi,v) - C(v)\Big) \int_{-\infty}^{\infty} \mathcal{E}_\eta(y_v)\, G(x,y,t;\xi,\eta,v)\, d\eta\, d\xi\, dv \\
&= (1-R)\mathcal{E}_y(y_t) \int_t^{T_m} \overline{\lambda}(v) B(t,v) \\
&\quad \Big( e^{x+I_\rho(t,v)-\frac{1}{2}I_1(0,t)}(1+k) D(v,T_m) F_2(v,T_m) \\
&\qquad N_2(a_1(x+I_\rho(t,v),t,v), b_1(x+I_\rho(t,v),t,v,T_m), R(t,v,T_m)) \\
&\quad - D(v,T_m) K N_2(a_2(x+I_\rho(t,v),t,v), b_2(x+I_\rho(t,v),t,v,T_m), R(t,v,T_m)) \\
&\quad - C(v) N(a_2(x+I_\rho(t,v),t,v))\Big) dv.
\end{aligned} \tag{52}
$$

Here in performing the integration w.r.t. $\xi$ we have followed the analogous calculation by Geske (1979) of the price of a compound call option on a call option.

This process can be continued to second order to obtain $h_2(x,y,t)$ but we do not consider such higher terms here. Setting $x=y=t=0$ in Eq. (52) and reverting to unscaled notation, we obtain Eq. (27). This concludes the proof of the theorem. $\square$

# B Proof of Theorem 4.1

We seek to derive Eq. (42) as the solution to Eq. (21) with the exposure function given by Eq. (41) for $v \in [t_j, t_{j+1})$. Let us take $h^{(j)}(x_t,y_t,t)$ for $t \le t_j$ as the value at time $t$ of the CVA contribution associated with defaults in the interval $[t_j,t_{j+1})$ and suppose it can be written

$$
h^{(j)}(x,y,t) = \epsilon h_1^{(j)}(x,y,t) + \epsilon^2 h_2^{(j)}(x,y,t) + O(\epsilon^3) \tag{53}
$$

for $O(1)$ functions $h_i^{(j)}(\cdot)$. The final condition satisfied is $h^{(j)}(x,y,t_{j+1}) = 0$. It is convenient to calculate the desired (first order) result $h_1^{(j)}(0,0,0)$ in two stages. First, estimate $h_1^{(j)}(x,y,t_j)$: as the underlying is *not* then a forward-starting option, the calculation is almost exactly as was performed before. Next we use the tower property of conditional expectations to assert that the CVA at time $t < t_j$ associated with the future period $[t_j,t_{j+1})$ is given by considering the same Eq. (21), but with the final condition that $h_1^{(j)}(x,y,t_j)$ is as previously determined. The desired result is then obtained by setting $t=0$.

Following the approach set out in Appendix A, we find from the first order calculation that

$$
h_1^{(j)}(x,y,t_j) = (1-R)\int_{t_j}^{t_{j+1}} \overline{\lambda}(v) \int_{\xi^*(x,v)}^{\infty} \Big(f_m(\xi,v) - C_j(x)\Big) \int_{-\infty}^{\infty} \mathcal{E}_\eta(y_v)\, G(x,y,t_j;\xi,\eta,v)\, d\eta\, d\xi\, dv, \tag{54}
$$

where

$$
\xi_j^*(x,v) := \inf\{\xi \mid f_m(\xi,v) - C_j(x) > 0\}.
$$

By its construction, we expect $\xi_j^*(x,v)$ to be approximately of the form

$$
\xi_j^*(x,v) = x + \xi_j^*(0,v). \tag{55}
$$

We will henceforth make this approximation, which simplifies the calculation considerably and adopt the convenient notation that $\xi_j^*(v) := \xi_j^*(0,v)$. On this basis we obtain

$$
h_1^{(j)}(x,y,t_j) \approx (1-R)\mathcal{E}_y(y_{t_j}) \int_{t_j}^{t_{j+1}} \frac{\overline{\lambda}(v)B(t_j,v)}{\sqrt{I_1(t_j,v)}} \int_{\xi_j^*(x,v)}^{\infty} N'\left(\frac{x+I_\rho(t_j,v)-\xi}{\sqrt{I_1(t_j,v)}}\right) f_m(\xi,v)\, d\xi\, dv \tag{56}
$$

$$
- (1-R)C_j(x)\mathcal{E}_y(y_{t_j}) \int_{t_j}^{t_{j+1}} \overline{\lambda}(v)B(t_j,v) N\left(\frac{I_\rho(t_j,v)-\xi_j^*(v)}{\sqrt{I_1(t_j,v)}}\right) dv \tag{57}
$$

Applying our Green's function straightforwardly to a payoff of $h_1^{(j)}(x,y,t_j)$ at final time $t_j$ we obtain

$$
\begin{aligned}
h_1^{(j)}(x,y,t) = & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,t;\xi,\eta,t_j)h_1^{(j)}(\xi,\eta,t_j)\,d\eta\,d\xi \\
& = (1-R)\mathcal{E}_y(y_t)\int_{t_j}^{t_{j+1}} \overline{\lambda}(v)B(t,v)D(v,T_m)\Big(e^{x+I_\rho(t,v)-\frac{1}{2}I_1(0,t)}(1+k)F_2(v,T_m) \\
& \qquad N_2(a_1^{(j)}(I_\rho(t,v),t,v),b_1(x+I_\rho(t,v),t,v,T_m),R(t,v,T_m)) \\
& \qquad - KN_2(a_2^{(j)}(I_\rho(t,v),t,v),b_2(x+I_\rho(t,v),t,v,T_m),R(t,v,T_m))\Big)dv \\
& - (1-R)\mathcal{E}_y(y_t)\int_{t_j}^{t_{j+1}} \overline{\lambda}(v)B(t,v)D(v,T_m)N\left(a_1^{(j)}(I_\rho(t_j,v),t_j,v)\right)dv\,\Big(e^{x+I_\rho(t,t_j)-\frac{1}{2}I_1(0,t)}F_2(t_j,T_m) \\
& \qquad N(b_1^{(j)}(x+I_\rho(t,t_j),t,T_m)) - KN(b_2^{(j)}(x+I_\rho(t,t_j),t,T_m))\Big)
\end{aligned} \tag{58}
$$

for $t < t_j$. Setting $x = y = t = 0$ and reverting to unscaled notation, we obtain Eq. (42). This concludes the proof of the theorem. □

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